The Critical Behavior for Percolation and the Mean Field Behavior for Oriented Percolation in High Dimensions

Lung-Chi Chen

Department of Mathematical Sciences, National Chengchi University

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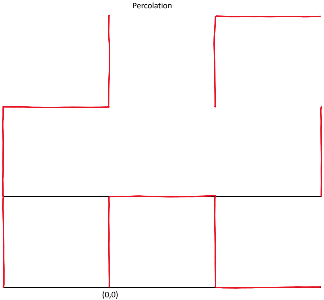
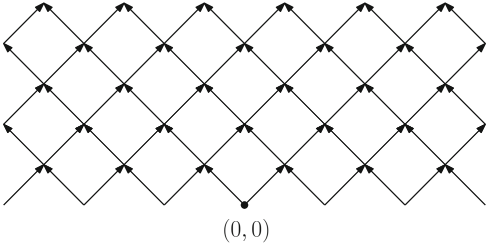
Percolation



Percolation

Oriented Percolation



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I. Percolation

Each bond $\{u, v\}$, which is a pair of vertices in $\mathbb{Z}^d$, is either occupied or vacant independently of the other bonds. The probability that $\{u, v\}$ is occupied is defined to be $pD(v - u)$, where $p \geq 0$ is the percolation parameter.

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II. Oriented Percolation

Each ordered bond $\{(u, t), (v, t + 1)\}$, which is a pair of vertices in $\mathbb{Z}^d \times \mathbb{Z}_+$, is either occupied or vacant independently of the other bonds. The probability that $\{(u, t), (v, t + 1)\}$ is occupied is defined to be $pD(v - u)$, where $p \geq 0$ is the percolation parameter.

The primary models in this talk are defined as follows:

1. Bernoulli models on $\mathbb{Z}^d$:

$$D(x) = \begin{cases} \frac{1}{2d}, & \text{if} \quad |x| = 1, \\ 0, & \text{others.} \end{cases}$$

The primary models in this talk are defined as follows:

1. Bernoulli models on $\mathbb{Z}^d$:

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Let $L \in \mathbb{N}$ fixed s.t.

$$D(x) = \frac{h(x/L)}{\sum_y h(y/L)},$$

2. spread-out models: $h \in (0, \infty)$ is symmetric with compact support. (e.q. $h(x) = 1_{\|x\|_\infty \le 1}$)

3. long-range spread-out models with order α: if $h \in (0, \infty)$ is symmetric function and $h(x) \approx |x|^{-d-\alpha}$ for some $\alpha > 0$.
($f(x) \approx g(x)$ means $|f(x)/g(x)| \in (0, \infty)$ as $|x| \to \infty$).

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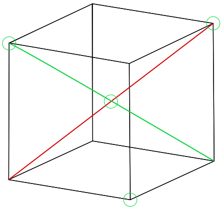
($f(x) \approx g(x)$ means $|f(x)/g(x)| \in (0, \infty)$ as $|x| \to \infty$).
4. Bernoulli models on $\mathbb{L}^d$ (Body Centered Cubic Lattice) :

$$D(x) = \begin{cases} \frac{1}{2^d}, & \text{if} \quad \prod_{j=1}^{d} |x_j| = 1, \\ 0, & \text{others.} \end{cases}$$



Let

$$C_o = \begin{cases} \{x \in \mathbb{Z}^d : o \leftrightarrow x\}, & \text{Percolation}, \\ \{x \in \mathbb{Z}^d : (o,0) \to (x,t) \text{ for some } t \in \mathbb{N}\}, & \text{OP}, \end{cases}$$

and let

$$\theta_p := P_p(|C_o| = \infty), \quad p_c(d) := \sup\{p : \theta_p = 0\}.$$

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Some well-known results:

- For Bernoulli percolation on $\mathbb{Z}^2$, Kesten (1980) showed that $p_c(2) = 2$.

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Some well-known results:

- For Bernoulli percolation on $\mathbb{Z}^2$, Kesten (1980) showed that $p_c(2) = 2$.However for Bernoulli oriented percolation on $\mathbb{Z}^2$, it is still open to find the value of $p_c(2)$.
- van den Berg and Keane (1982) showed that θ_p is continuous on $(p_c(d), 1/\|D\|_\infty]$.
- Burton and Keane (1989) showed that If $\theta_p > 0$, then $P_p(\exists$ a unique infinite cluster $) = 1$.

Let

$$\chi_p := E_p(|C_0|),\ p < p_c, \quad \varphi_p(x) = P_p(o \leftrightarrow x),$$
$$\xi_p^{-1} = \lim_{n\to\infty}\{-\frac{1}{n}\log P_p(o \leftrightarrow \partial B(n))\}, \quad p < p_c,$$

where χ_p is called the susceptibility, and ξ_p is called the correlations length.

critical exponents for percolation

Let

$$\chi_p := E_p(|C_0|),\ p < p_c, \quad \varphi_p(x) = P_p(o \leftrightarrow x),$$
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where χ_p is called the susceptibility, and ξ_p is called the correlations length.
Note that if we definition

$$\tau_p = \sqrt{\sum_{x\in\mathbb{Z}^d} |x|^2 \varphi_p(x)},\ p < p_c.\ (\text{ mean square displacement})$$

It can be showed that $\xi_p \approx \tau_p$ as p is near p_c.

For finite-range percolation models, it is expected that, around $p = p_c$, we observe the following power behavior characterized by the critical exponents β, γ, η, ν and δ such that

$$\theta_p \approx_{p \downarrow p_c} (p - p_c)^{\beta}, \quad \chi_p \approx_{p \uparrow p_c} (p_c - p)^{-\gamma},$$
$$\varphi_{p_c}(x) \approx |x|^{2-d-\eta}, \quad \xi_p \approx_{p \uparrow p_c} (p - p_c)^{\nu},$$
$$P_{p_c}(|C_0| = n) \approx n^{-1-1/\delta}.$$

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For long-range percolation models, we modify τ_p as follows:

$$\tau_p = \Big(\sum_{x\in\mathbb{Z}^d} |x|^r \varphi_p(x)\Big)^r, \quad p < p_c, \quad r < \alpha.\text{(Gyration radius)}$$

It is expected that, around $p = p_c$, we observe the following power behavior characterized by the critical exponents $\beta, \gamma, \delta, \ldots$ such that

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$$\varphi_{p_c}(x) \approx |x|^{(\alpha\wedge 2)-d-\eta}, \quad \xi_p \approx_{p\uparrow p_c} (p - p_c)^{\nu},$$
$$P_{p_c}(|C_0| = n) \approx n^{-1-1/\delta}.$$

The exponents γ, η, β and ν are predicted to be Fisher's relations as follows:

$$\gamma = (\alpha \wedge 2 - \eta)\nu, \quad \gamma + 2\beta = \beta(\delta + 1).$$

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Question: What are the values of γ, η, β and ν ?

For finite-range percolation models

	d=2	d=3	d=4	d=5	d=6	d>6
β	5/36 0.14 (1974)	0.4053 (2021)	0.6590 (2015)	0.8457 (2015)		1
γ	43/18	1.819 (2021)	1.45 (2015)	1.1817 (2015)		1
δ	91/5 18 (1980)	5.29 (1998)	3.198 (1997)	3.0 (1980)		2
η	5/24	-0.047 (2015)	-0.0954 (2015)	-0.0565 (2015)		0
ν	4/3 1.33(1975)	0.8774 (2016)	0.6845 (2021)	0.5737 (2021)		1/2

Let

$$\nabla_p(w) = \sum_{x,y\in\mathbb{Z}^d} \varphi_p(x)\varphi_p(y-x)\varphi_p(w-y)$$

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Theorem (Aizenman and Newman (1984), Barsky and Aizenman (1991))

The triangle condition implies $\gamma = 1$, $\beta = 1$ and $\nu = \frac{1}{2}$. (Mean fields behavior)

Theorem (Hara and Slade, 1990) For spread-out model with $L \gg$ and $d > 6$, we have

$$\gamma = 1, \nu = \frac{1}{2} \quad \text{for any dimensions } d > 6.$$

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Theorem (Chen and Sakai, 2015) For long-range model with $\alpha \neq 2$ and $L \gg$, we have

$$\gamma = 1,\, \nu = \frac{1}{\alpha \wedge 2} \text{ and } \eta = 0 \quad \text{for any dimensions } d > 3(\alpha \wedge 2).$$

Theorem (Chen and Sakai, 2019) For long-range model with $\alpha = 2$ and $L \gg$, we have

$$\gamma = 1,\ \nu = \frac{1}{\alpha \wedge 2} \text{ and } \eta = 0 \quad \text{with logarithmic terms}$$

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for any dimensions $d \geq 6$.

Theorem (Handa, Kamijima and Sakai (2019) For Bernoulli percolation on d-dimensional BCC lattice with $d \geq 9$, we have

$$\gamma = 1, \quad \nu = \frac{1}{2}.$$

critical exponents for oriented percolation

Let

$$\chi_p := E_p(|C_0|), \quad \varphi_p(x,n) = P_p((o,0) \to (x,n)),$$
$$\tau_p(n) := \sqrt{\sum_{x \in \mathbb{Z}^d} |x|^2 \varphi_p(x,n)}, \quad p < p_c,$$

where χ_p is called the susceptibility, ξ_p is called the correlations length and $\tau_p(n)$ is called the mean square displacement at time n.

For finite-range oriented percolation models, it is expected that, around $p = p_c$, we observe the following power behavior characterized by the critical exponents $\beta, \gamma, \delta, \ldots$ such that

$$\theta_p \approx_{p \downarrow p_c} (p - p_c)^{\beta}, \quad \chi_p \approx_{p \uparrow p_c} (p_c - p)^{-\gamma},$$
$$\sum_{n=0}^{\infty} \varphi_{p_c}(x, n) \approx |x|^{2-d-\eta}, \quad \tau_{p_c}(n) \approx n^{\nu}$$
$$P_{p_c}(|C_0| = n) \approx n^{-1-1/\delta}.$$

	d=1	d=2	d=3	d=4	d>4	
β	0.276 (2013)	0.580 (2013)	0.818 (2013)		1	
γ	2.277730 (1999)	1.595 (1992)	1.237 (1992)		1	
δ	0.15944 (2013)	0.451 (2013)	0.7398 (2013)		1	
η	0.31370 (2013)	0.2307 (2013)	0.1057 (2013)		0	
ν	1.0979 (2013)	0.729 (2013)	0.582 (2013)		1/2	

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$$\tau_p(n) = \Big(\sum_{x\in\mathbb{Z}^d} |x|^r \varphi_p(x,n)\Big)^r, \quad p < p_c, \quad r < \alpha.$$

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&\sum_{n=0}^{\infty} \varphi_{p_c}(x,n) \approx |x|^{\alpha\wedge 2-d-\eta}, \quad \tau_{p_c}(n) \approx n^{\nu} \\
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\end{aligned}$$

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Theorem (Chen and Sakai, 2008, 209) For long-range model with $\alpha > 0$ and $L \gg$, we have

$$\gamma = 1,\, \nu = \frac{1}{\alpha \wedge 2} \quad \text{for any dimensions } d > 2(\alpha \wedge 2).$$

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Theorem (Chen, Handa and Kamijima, submitted) For Bernoulli model on d-dimensional BCC lattices, we have

$$\gamma = 1,\ \nu = \frac{1}{\alpha \wedge 2} \quad \text{for any dimensions } d \geq 9.$$

Key arguments

1. Lace expansion and Bootstrapping argument (OP models)
By lace expansion

$$\varphi_p(x,t) = \delta_{o,x}\delta_{0,t} + \Pi_p^N(x,t) + \Big((\delta_{o,\cdot}\delta_{0,\cdot} + \Pi_p^N) * q_p * \varphi_p\Big)(x,n) + (-1)^{N+1} R_p^{N+1}(x,t),$$

where

$$\Pi_p^N(x,t) = \sum_{n=0}^{N} (-1)^n \pi_p^{(n)}(x,t), \quad q_p(x,t) = D(x)\delta_{t,1}.$$

We can show that for $p < p_c$, we get $|R_p^N(x,t)| \to 0$ for all (x,t) then

$$\varphi_p(x,t) = \delta_0(x)\delta_{0,t} + \Pi_p(x,t) + \Big((\delta_{o,\cdot}\delta_{0,\cdot} + \Pi_p) * q_p * \varphi_p\Big)(x,n)$$

Define

$$
\begin{aligned}
g_1(p,m) &:= p(m\vee 1),\\
g_2(p,m) &:= \sup_{k\in[-\pi,\pi]^d\; z\in\mathbb{C},\,|z|\in(1,m)} \frac{|\varphi_p(k,z)|}{|\hat{S}_{\mu_p(z)}(k)|},\\
g_3(p,m) &:= \sup_{k\in[-\pi,\pi]^d\; z\in\mathbb{C},\,|z|\in(1,m)} \frac{|\frac{1}{2}\nabla_k(\hat{q}_p(l,z)\hat{\varphi}_p(l,z)|}{\hat{U}_{\mu_p(z)}(k,l)},
\end{aligned}
$$

where $\nabla_k \hat{f}(l) = \hat{f}(l+k) + \hat{f}(l-k) - 2\hat{f}(k)$ and $\mu_p(z) = \left(1 - \hat{\varphi}_p(o,|z|)^{-1}\right) e^{i\,argz}$. For the nearest-neighbor oriented percolation on BCC lattices with dimensions $d \geq 8$, we can show that $\{g_i(p,m)\}_{i=1}^3$ satisfies the following three properties:

- $\{g_i(p,m)\}_{i=1}^3$ are continuous in $m \in [0, m_p)$ for every $p \in (0, p_c)$.
- There are finite constant $\{K_i\}_{i=1}^3$ such that $g_i(0,1) < K_i$ with $K_i > 1$ for $i = 1, 2, 3$.
- We fix both $p \in (0, p_c)$ and $z \in \mathbb{C}$ with $|z| \in (1, m_p)$ and assume $g_i(p,m) \leq K_i,\ i = 1, 2, 3$. Then, the stronger inequalities $g_i(p,m) < K_i,\ i = 1, 2, 3,$ hold.

function $\varphi_p(x,t)$ by a line segment. For example,

$\varphi_p(\boldsymbol{x}) =$  , $(q_p^{\star 2} \star \varphi_p)(\boldsymbol{x}) =$  ,

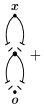
$B^{(1,2)}_{p,m} = \sup_{\boldsymbol{x}}$  , $T^{(2,1)}_{p,m} = \sup_{\boldsymbol{x}}$  , $\hat{V}^{(2,2)}_{p,m}(k) = \sup_{\boldsymbol{x}}$  ,

where the unlabeled vertices (short lines and dots) are summed over $\mathbb{L}^d \times \mathbb{Z}_+$. The segments emphasized by the braces mean weighted two-point functions or weighted space-time transition probabilities: φ_p or q_p multiplied by m's or $1-\cos k\cdot\bullet$. Time increases from the beginning point to the ending point of a line.

Lemma B.1. *For $\boldsymbol{x} \in \mathbb{L}^d \times \mathbb{Z}_+$ and $N \geq 2$,*

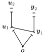
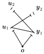
$\left|\pi_p^{(0)}(\boldsymbol{x}) - \pi_p^{(1)}(\boldsymbol{x})\right|$

$\leq \frac{1}{2} \times$  $+$  $+ \frac{3}{2} \times$  $+ 3 \times$  $+ 2 \times$  $+$  , (B.1)

$\pi_p^{(2)}(\boldsymbol{x}) \leq$  $+ 2 \times$  $+$  $+$ 

$+ \frac{1}{2} \times$  $+$  $+ \frac{1}{2} \times$  $+$  , (B.2)

$\pi_p^{(N)}(\boldsymbol{x}) \leq \sum_{\substack{\{\boldsymbol{y}_i\}_{i=1}^N, \{\boldsymbol{u}_i\}_{i=1}^N \\ (\forall i:\, \tau(\boldsymbol{y}_i) > \tau(\boldsymbol{u}_{i-1}))}} \Bigg(2\delta_{\boldsymbol{u}_1,\boldsymbol{o}}\delta_{\boldsymbol{y}_1,\boldsymbol{o}} \times$  $+ \frac{1}{2}\delta_{\boldsymbol{y}_1,\boldsymbol{o}} \times$ 

$+$  $+ \frac{1}{2}\delta_{\boldsymbol{y}_1,\boldsymbol{o}} \times$  $+$  $\Bigg)$ (B.3)

$\times \prod_{i=2}^{N-1} \Bigg($  $+$  $\Bigg) \times$  .

To estimate the upper bound of $g_2(p, m)$, we need to estimate the upper bound as follows

$$\frac{\hat{\Pi}_p(o, |z|) - \hat{\Pi}_p(k, |z|)}{|1 - e^{i\theta}\hat{D}(k)|}, \quad \text{where } z = re^{i\theta}, \theta \in (-\pi, \pi). \; r \in [1, m_p).$$

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Proposition For BCC lattice $\mathbb{L}^d$ with $d \geq 2$, we have

$$\begin{aligned} g_2(p,m) \quad &:= \sup_{k\in[-\pi,\pi]^d\ z\in\mathbb{C},\,|z|\in(1,m_p)} \frac{|\varphi_p(k,z)|}{|\hat{S}_{\mu_p(z)}(k)|}, \\ &= \sup_{k\in[-\pi/2,\pi/2]^d\ z\in\mathbb{C},\,|z|\in(1,m_p)} \frac{|\varphi_p(k,z)|}{|\hat{S}_{\mu_p(z)}(k)|}, \end{aligned}$$

2. Triangle condition

a. For percolation models, let $G(x) := \sum_{n=0}^{\infty} S_n(x) := 1 + \sum_{n=1}^{\infty} \hat{D}^{*n}(x)$. Then using inverse formula, we get

$$\nabla_{p_c} = \sum_{x \in \mathbb{Z}^d} G_{p_c}^{*3}(x) = \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} \hat{G}_{p_c}(k)^3$$

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$$\begin{aligned} \nabla_{p_c} &= \sum_{x \in \mathbb{Z}^d} G_{p_c}^{*3}(x) = \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} \hat{G}_{p_c}(k)^3 \\ &\approx \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} \frac{1}{\left(1 - \hat{D}(k)\right)^3}. \end{aligned}$$

b. For oriented percolation models, let $G(x,n) := S_n(x) := \sum_{n=1}^{\infty} \hat{D}^{*n}(x)$. Then using the Hausdorff-Young inequality, we get

$$\begin{aligned} \sup_{(x,n)} \nabla_p(w,n) &\leq \sup_{(x,n)} \sum_{(w,s)} \varphi_p^{*2}(w,s)\varphi_p(w-x,s-n) \\ &\leq \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} \left|\hat{\varphi}_{p_c}(k,e^{i\theta})^2 \hat{\varphi}_{p_c}(k,e^{-i\theta})\right| \end{aligned}$$

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Using the infrared bound (i.e., $|\hat{\varphi}_{p_c}(k,e^{i\theta})| \le \frac{K}{|1-e^{i\theta}\hat{D}(k)|}$ for some $K \in \mathbb{R}$), Then

$$\sup_{(x,n)} \nabla_p(w,n) \approx \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} \frac{e^{-ik\cdot x}}{\left|1-\hat{D}(k)e^{i\theta}\right|^3} < \infty, \quad d > 8.$$

Then triangle condition holds.